$$P(x|\theta) = \int P(x|z,\theta)\,P(z)\,dz$$

$$\mathcal{L}(\theta|D) = \sum_{i=1}^{n} \log P(x_i|\theta)$$

Approximate the log likelihood

Kullback-Leibler Divergence

$$KL(q(x)|P(x)) = \int q(x)\log\frac{q(x)}{P(x)}\,dx = \underbrace{\int q(x)\log q(x)\,dx}_{E_{x\sim q}[\log q(x)]} - \underbrace{\int q(x)\log P(x)\,dx}_{E_{x\sim q}[\log P(x)]}$$

$$KL(q(x)|P(x)) \neq KL(P(x)|q(x))$$

$$KL(q(z|x)|P(z|x)) = \int q(z|x)\log\frac{q(z|x)}{P(z|x)}\,dz \qquad P(z|x) \longrightarrow \frac{P(x|z)\,P(z)}{P(x)}$$

$$= \int q(z|x)\log\frac{P(x)\,q(z|x)}{P(x|z)\,P(z)}$$

$$= \int q(z|x)\left[\log P(x) + \log q(z|x) - \log P(x|z) - \log P(z)\right]$$

$$KL\left(q(z|x) \,|\, P(z|x)\right)$$

$$= \int q(z|x)\left[\log P(x) + \log q(z|x) - \log P(x|z) - \log P(z)\right] dz$$

$$= \log P(x) + \underset{z\sim q}{E}\left[\log q(z|x)\right] - \underset{z\sim q}{E}\left[\log P(x|z)\right] - \underset{z\sim q}{E}\left[\log P(z)\right]$$

$$KL\left(q(z|x) \,|\, P(z)\right)$$

$$\log P(x) = E\left[\log P(x|z)\right] - \underbrace{KL\left(q(z|x) \,|\, P(z|x)\right)}_{\geq 0} - KL\left(q(z|x) \,|\, P(z)\right)$$

$$\log P(x) \geq E\left[\log\left(P(x|z)\right)\right] - KL\left(q(z|x) \,|\, P(z)\right)$$

ELBO : Expected Lower BOund

$$\log P(x) \geq E[\log(P(x|z))] - KL(q(z|x) \| P(z))$$

$$\text{Loss} \doteq -\log P(x) = KL(\underbrace{q(z|x)}_{N(\mu,\, \mathrm{diag}(\sigma^2))} \| \underbrace{P(z)}_{N(0,\, I)}) - \underbrace{E[\log(P(x|z))]}_{\text{Reconstruction Error}},$$

$$\mu, \sigma^2 \in \mathbb{R}^M$$

$$KL(q(z|x) \| P(z)) = \frac{1}{2} \sum_{i=1}^{M} 1 + \ln \sigma_i^2(x) - \mu_i^2(x) - \sigma_i^2(x)$$

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$$\varepsilon \sim N(0, 1)$$

$$z_i = \mu_i + \sigma_i \varepsilon \sim N(\mu_i, \sigma_i^2)$$